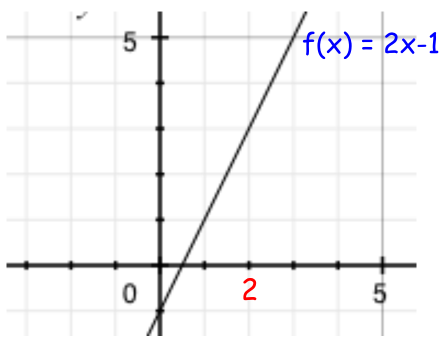
Introduction to Limits

Limit - as the x-value approaches some value, the y-values get close to a value.



$$\lim_{x \to 2} 2x - 1 = ?$$

" *What is the limit of 2x-1 as x approaches 2 ?* "

x	1.00	1.50	1.60	1.90	1.99	1.999		2.001	2.01	2.10	2.50
f(x)	1.00	2.00	2.20	2.80	2.98	2.998		3.002	3.02	3.20	4.00

Limit Definitions

- Given the real numbers *L* and *c*,

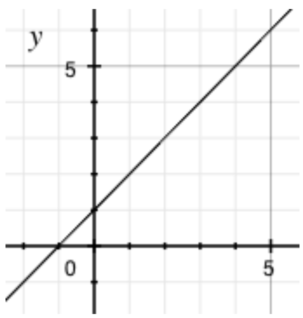
1. $\lim_{x \to c} f(x) = L$ "The limit as x approaches *c* is *L*" *General Limit

2. $\lim_{x \to c^+} f(x) = L_1$ "The limit as x approaches *c from the right* is L_1" *Right-Hand Limit

3. $\lim_{x \to c^-} f(x) = L_2$ "The limit as x approaches *c from the left* is L_2" *Left-Hand Limit

Note: In general for any function f(x), if $L_1=L_2$, then the limit of f(x) = L, ..if $L_1 \neq L_2$, then the limit does not exist, DNE.

Ex)

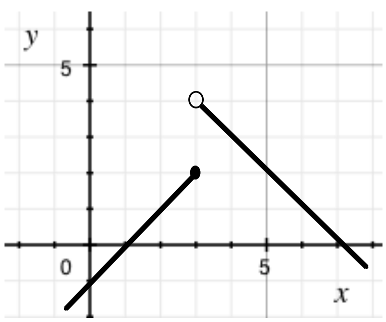


$$\lim_{x \to 2^+} f(x) =$$

$$\lim_{x \to 2^-} f(x) =$$

$$\lim_{x \to 2} f(x) =$$

Ex)

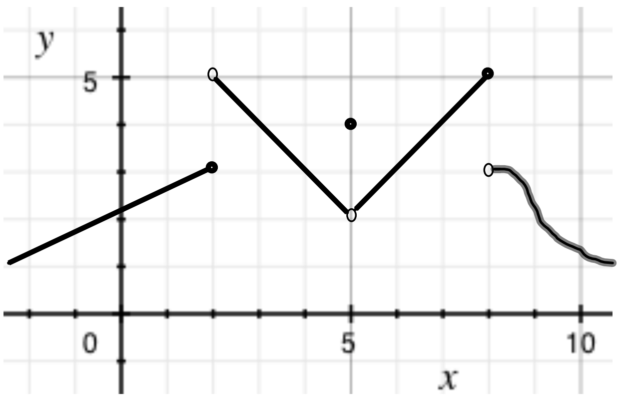


$$\lim_{x \to 3^+} f(x) =$$

$$\lim_{x \to 3^-} f(x) =$$

$$\lim_{x \to 3} f(x) =$$

Ex)



1) $\lim_{x \to 2^+} f(x) =$

$\lim_{x \to 2^-} f(x) =$

$\lim_{x \to 2} f(x) =$

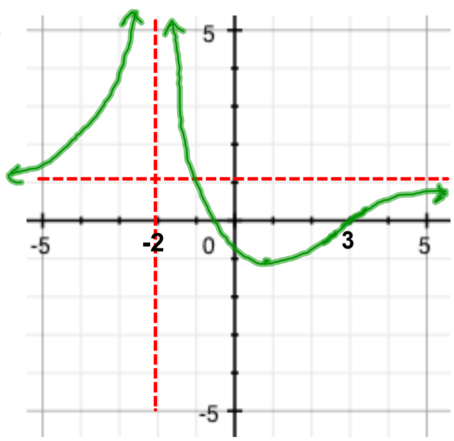
2) $\lim_{x \to 5^+} f(x) =$

$\lim_{x \to 5^-} f(x) =$

$\lim_{x \to 5} f(x) =$

3) *Does the limit exist at x = 8?*

Ex)



$$\lim_{x \to -2^+} f(x) =$$

$$\lim_{x \to -2^-} f(x) =$$

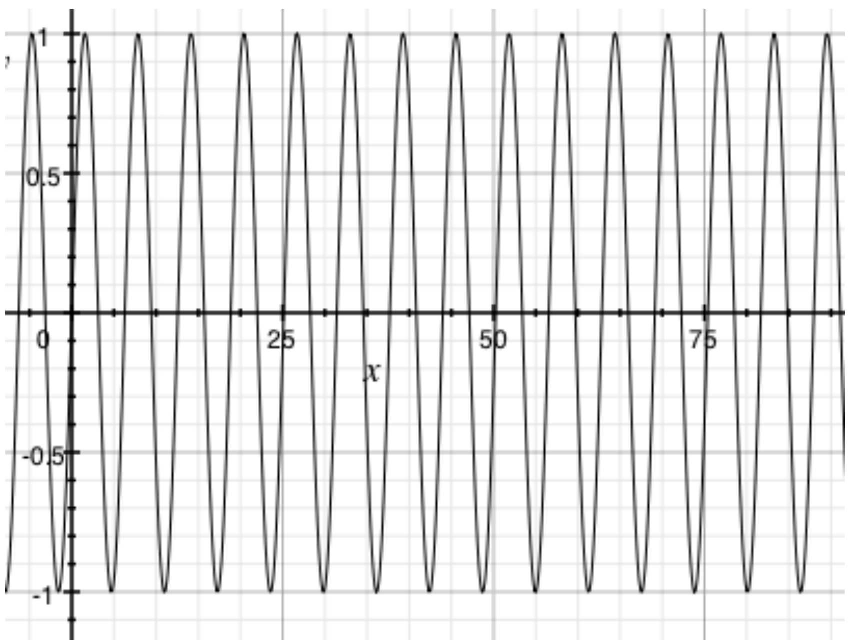
$$\lim_{x \to -2} f(x) =$$

$$\lim_{x \to 3} f(x) =$$

$$\lim_{x \to \infty} f(x) =$$

* Limits fail to exist (DNE) because of **Oscillation**.

$$\lim_{x \to \infty} \sin(x) = DNE$$



Solving Limits Algebraically

Rules: 1) If f is a constant function f(x) = k, then

$$\lim_{x \to c} f(x) = \lim_{x \to c} k = k$$

2) If f is a identity function f(x) = x, then

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c$$

3) $$\lim_{x \to c} \left[f(x) \pm g(x) \right] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$$

4) $$\lim_{x \to c}\left[f(x) \bullet g(x)\right] = \lim_{x \to c} f(x) \bullet \lim_{x \to c} g(x)$$

5) $$\lim_{x \to c}\left[f(x) / g(x)\right] = \lim_{x \to c} f(x) / \lim_{x \to c} g(x)$$

* *where* $\lim_{x \to c} g(x) \neq 0$

6) $$\lim_{x \to c} \sqrt[n]{f(x) =} \quad \sqrt[n]{\lim_{x \to c} f(x)}$$

Computing Limits Algebraically

1. Substitute *c* in for *x*, $\lim_{x \to c} f(x) = f(c)$

2. Fractional Limits

• if $\lim_{x \to c} \frac{h(x)}{g(x)}$ $gives$ $\frac{0}{0}$, the limit **may** or **may not** exist

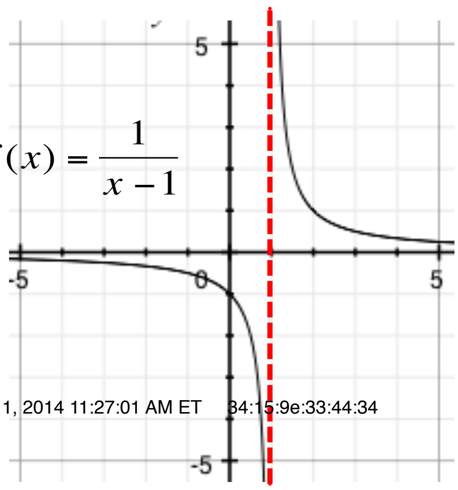
* Factor/Simplify -> Substitute again
* w/Radicals -> Multiply by conjugate -> Substitute again
* Graph

• if $\lim_{x \to c} \frac{h(x)}{g(x)}$ $gives$ $\frac{0}{\#}$, the limit is 0

• if $\lim_{x \to c} \frac{h(x)}{g(x)}$ $gives$ $\frac{\#}{0}$, the limit DNE ($+\infty$ / $-\infty$

Ex)

$$f(x) = \frac{1}{x-1}$$



$$\lim_{x \to 1^+} f(x) =$$

$$\lim_{x \to 1^-} f(x) =$$

$$\lim_{x \to 1} f(x) =$$

*** What to do?**

1) Pick # close to "break point" from correct side
2) Check signs of numerator and denominator

★ if signs the same $\frac{+}{+}$ *or* $\frac{-}{-}$, lim = ∞

★ if signs different $\frac{+}{-}$ *or* $\frac{-}{+}$, lim = $-\infty$

Examples)

1.) $\lim_{x \to 3} x + 4 \ =$

2.) $\lim_{x \to 4} \frac{2x}{x + 4} \ =$

3.) $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 1} \ =$

4.) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \ =$

5.) $\lim_{x \to 4} \frac{2x}{x-4} =$

6a.) $\lim_{x \to 2^+} \frac{x}{x-2} =$

6b.) $\lim_{x \to 2^-} \frac{x}{x-2} =$

6c.) $\lim_{x \to 2} \frac{x}{x-2} =$

Limits approaching $\pm\infty$

*** Polynomials:**

- limit behaves like it's term of highest degree

To Do: 1) Find highest degree
2) Substitute # "close" to ∞ or $-\infty$,

- If end result is +, lim is ∞
- If end result is -, lim is $-\infty$

Ex) $\lim_{x \to \infty} 7x^5 - 4x^3 + 2x - 9 =$

Ex) $\lim_{x \to -\infty} 17x^5 - 4x^8 - 5x + 1 =$

$$Ex)\quad \lim_{x \to \infty} -4x^4 + 10x^3 + 100 =$$

$$Ex)\quad \lim_{x \to -\infty} -3x^5 + x^2 =$$

* Rational Expressions

- behaves like term of highest degree

To Do: 1) Find highest degree in numerator
2) Find highest degree in denominator

• if degree of numerator < degree of denominator, ***then the limit is zero.***

• if degree of numerator = degree of denominator, **then the limit is** $\frac{a}{b}$, where ***a*** and ***b*** are the leading coefficients of the higher degrees.

- if degree of numerator > degree of denominator, We need to substitute a number "close" to $\pm\infty$ and evaluate signs;

$\frac{+}{+}$ *or* $\frac{-}{-}$ then the limit is ∞

$\frac{+}{-}$ *or* $\frac{-}{+}$ then the limit is $-\infty$

Examples) 1) $\lim_{x \to \infty} \frac{3 - x}{x^2 + 45}$

2) $\lim_{x \to \infty} \frac{x^3 - 4x^2 + 7}{3 - 6x - 2x^3}$

3) $\lim_{x \to -\infty} \frac{x^2 + 6x + 8}{x^2 - 5x - 14}$

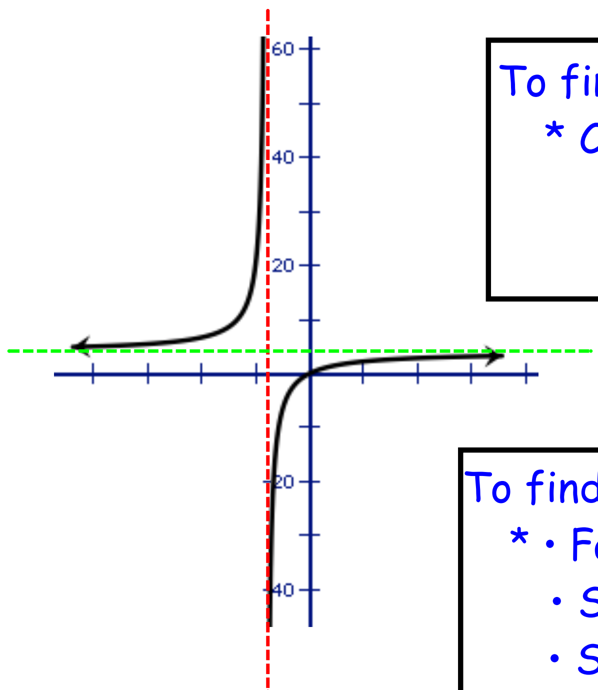
4) $\lim_{x \to -\infty} \frac{x^3 - 5}{1 + x^2}$

5) $\lim_{x \to \infty} \frac{3x^2 - 6x}{4x - 8}$

6) $\lim_{x \to \infty} \frac{x}{|x|}$

7) $\lim_{x \to -\infty} \frac{\sqrt{x^2 + 2}}{3x - 8}$

* *The graphs of some functions seem to approach some vertical or horizontal line, such lines are called* <u>*Asymptotes*</u>.



Horizontal Asymptote : y = b

Vertical Asymptote : x = a

To find Horizontal Asymptotes:

* Compute $\lim_{x \to \infty} f(x) = c$

so, **y = c**

To find Vertical Asymptotes:

* • Factor numerator and denominator,

• Simplify

• Set denominator = to 0, Solve for a

so, **x = a**

•In each of the following, find the Horizontal and vertical asymptotes, (HA's and VA's), *if they exist*.

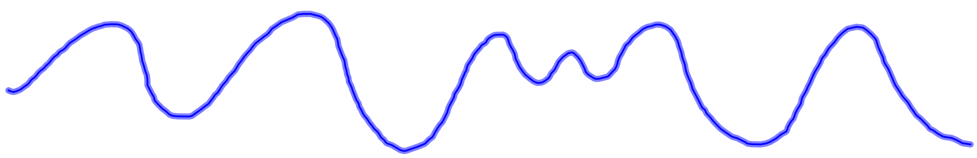
1.) $f(x) = \dfrac{x-6}{x^2-16}$

2.) $f(x) = \dfrac{x-4}{x^2-16}$

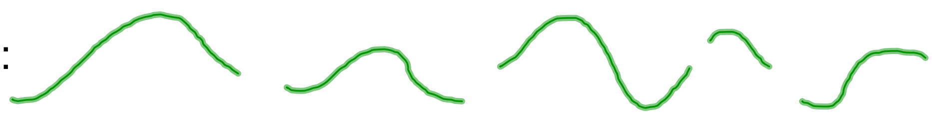
Continuity: w/ Functions

*If you can draw a function without lifting up your pencil off the paper, then the function is <u>*Continuous!*</u>

Continuous:



Discontinuous:



Definition of Continuity at a Point

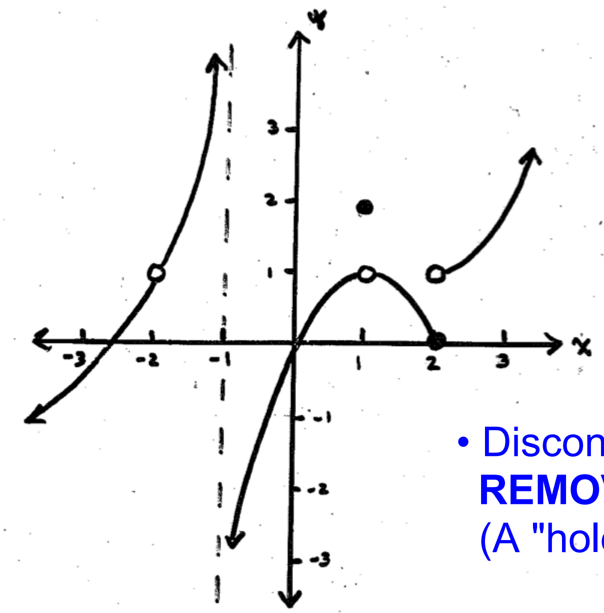
A function $f(x)$ is said to be continuous at a point $x = a$ if;

1) $f(a)$ exists

2) $\lim_{x \to a} f(x)$ exists

3. $f(a) = \lim_{x \to a} f(x)$

ALL 3 MUST BE TRUE!

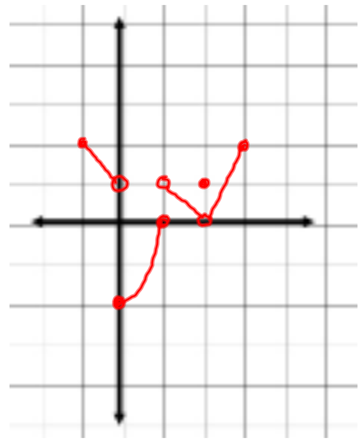


f(x) is discontinuous
at x = -2, -1, 1, 2

• Discontinuities at x = -2 and 1 are called
REMOVABLE DISCONTINUITIES.
(A "hole" or a "hole" w/extra point)

• Discontinuities at x = -1 and 2 are called
NON-REMOVABLE DISCONTINUITIES.
(A "break" "skip", or a "gap")

Ex) Given $f(x) = \begin{cases} 1 - x, & -1 \le x < 0 \\ 2x^2 - 2, & 0 \le x \le 1 \\ -x + 2, & 1 < x < 2 \\ 1, & x = 2 \\ 2x - 4, & 2 < x \le 3 \end{cases}$



Given the graph of *f(x)*

a) Where is f(x) discontinuous?
Prove using Definition of Continuity.

b) Where does f(x) have : 1) removable discontinuity
2) non-removable discontinuity

Ex) Where, if any, is $f(x) = \dfrac{x}{x^2 + 3x + 2}$ *discontinuous?*

Ex) Determine the value of *k* such that *f(x)* is a continuous when,

$$f(x)=\begin{cases}3kx-5, & x>2\\ 4x-5k, & x\leq 2\end{cases}$$

Think! f(x) needs to "hook up" at x = 2

Ex) Determine the value of *k* such that *f(x)* is a continuous when,

$$f(x)=\begin{cases}kx-1, & x<2\\ kx^2, & x\geq 2\end{cases}$$

Ex) Given f(x), solve for *a* and *b* to make f(x) continuous.

$$f(x)=\begin{cases} 6x+12, & x \le -2 \\ ax^3+b, & -2 < x < 1 \\ 2x+\dfrac{5}{2}, & x \ge 1 \end{cases}$$

Limits of Trigonometric Functions

Rules:

1) $\lim_{\Theta \to 0} \sin\Theta = 0$

2) $\lim_{\Theta \to 0} \cos\Theta = 1$

3) $\lim_{x \to 0} \frac{\sin ax}{bx} = \lim_{x \to 0} \frac{ax}{\sin bx} = \lim_{x \to 0} \frac{\sin ax}{\sin bx} = \frac{a}{b}$

4) $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$

5) $\lim_{x \to \pm\infty} \sin x = dne$

6) $\lim_{x \to \pm\infty} \cos x = dne$

$Ex1)\quad \lim_{x\to 0} \frac{\sin 3x}{7x} =$

$Ex2)\quad \lim_{x\to 0} \frac{\sin 6x}{\sin 11x}$

$Ex3)\quad \lim_{x\to 0} \tan x$

$Ex4)\quad \lim_{x\to \Pi} \cos x$

$Ex5)\quad \lim_{x\to 0} \frac{3}{\cos x}$

$Ex6)\quad \lim_{x\to 0} x\cot x$

$$Ex7)\quad \lim_{x \to 0} \frac{(\sin x)^4}{x^3} \quad =$$

$$Ex8)\quad \lim_{x \to 0} \frac{1-\cos^2 x}{x}$$

$$Ex9)\quad \lim_{x \to 0} \frac{1-\cos 2x}{\sin 3x}$$